Exercise 1. Let $C^0_{2\pi}(\mathbb{R},\mathbb{R}^3)$ be the space of continuous 2π-periodic functions. The Douglas functional is defined for all $f \in C^0_{2\pi}(\mathbb{R},\mathbb{R}^3)$ by

$$D(f) = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{2\pi}\frac{|f(\theta)-f(\varphi)|^2}{4\sin^2\left(\frac{\theta-\varphi}{2}\right)}d\theta\,d\varphi,$$

coincides with the Dirichlet energy of its harmonic extension (to be defined below)

$$E(u) = \frac{1}{2}\int_{\mathbb{D}}|\nabla u|^2dx.$$

1. Show that

$$D(f) = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{2\pi}\frac{|f(\theta)-f(\varphi)|^2}{|e^{i\theta}-e^{i\varphi}|^2}d\theta\,d\varphi.$$

2. Using the Poisson formula (or the theory of Fourier series), show that if $f \in C^0(\mathbb{R},\mathbb{R}^3)$ is a 2π-periodic function, the only function $u \in C^2(\mathbb{D},\mathbb{R}^3)\cap C^0(\overline{\mathbb{D}},\mathbb{R}^3)$ such that

$$\begin{cases}\Delta u = 0 & \text{in } \mathbb{D}\\ u(e^{i\theta}) = f(\theta) & \forall\ \theta\in\mathbb{R}\end{cases}$$

is given by

$$u(re^{i\theta}) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} r^n\left(a_n\cos(n\theta)+b_n\sin(n\theta)\right),$$

where $a_n = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} f(\theta)\cos(n\theta)d\theta$ and $b_n = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} f(\theta)\sin(n\theta)d\theta$.

We say that u is the *harmonic extension* of f.

3. Show that for all $n \geq 1$, we have

$$\begin{aligned}|a_n|^2+|b_n|^2 &= \frac{1}{\pi^2}\int_0^{2\pi}\int_0^{2\pi} f(\theta)f(\varphi)\cos(n(\theta-\varphi))d\theta\,d\varphi\\ &= -\frac{1}{2\pi^2}\int_0^{2\pi}\int_0^{2\pi}|f(\theta)-f(\varphi)|^2\cos(n(\theta-\varphi))d\theta\,d\varphi\end{aligned}$$

4. Show that for all $0<r<1$, we have

$$E_r(u) = \frac{1}{2}\int_{B(0,r)}|\nabla u|^2dx = \frac{\pi}{2}\sum_{n=1}^{\infty} n\,r^{2n}\left(|a_n|^2+|b_n|^2\right).$$

5. Define

$$Q(r,\alpha) = \begin{cases}-\displaystyle\sum_{n=1}^{\infty} n\,r^{2n}\cos(n\alpha) & \text{for all } 0\leq r<1\\ \dfrac{1}{4\sin^2\left(\frac{\alpha}{2}\right)} & \text{for } r=1\end{cases}$$

and deduce that

$$E_r(u) = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{2\pi}Q(r,\theta-\varphi)|f(\theta)-f(\varphi)|^2d\theta\,d\varphi \quad \text{for all } 0<r<1.$$

6. Show that $Q(r,\alpha) \leq Q(1,\alpha)$ for all $0 \leq r < 1$ and that $Q(r,\alpha) \to Q(1,\alpha)$ for all $\alpha \neq 0 \pmod{2\pi}$.
7. Assuming that $D(f) < \infty$, deduce that
$$E_r(u) \underset{r\to 1}{\longrightarrow} E(u) = D(f).$$
8. Conversely, assume that $E(u) < \infty$ and for all $0 < \varepsilon < \pi$, let
$$R(\varepsilon) = [0,2\pi] \times [0,2\pi] \cap \left\{(\theta,\varphi) : |e^{i\theta} - e^{i\varphi}| > \varepsilon\right\}.$$
Show that for all $0 < \varepsilon < \pi$, we have
$$\int_{R(\varepsilon)} Q(1,\theta-\varphi)|f(\theta)-f(\varphi)|^2 d\theta\, d\varphi \leq 4\pi\, E(u) < \infty.$$
9. Deduce that $D(f) < \infty$ and conclude the proof of the theorem.